Solutions of Exercises from 'A Short Introduction to Quantum Information and Quantum Computation'

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Chapter 2

Exercises from Chapter 2

2.6.1 Determination of the polarization of a light wave

1. We can choose $\delta_x = 0$, $\delta_y = \delta$. The equation of the ellipse

$$x = \cos\theta\,\cos\omega t \qquad y = \sin\theta\,\cos(\omega t - \delta)$$

reads in Cartesian coordinates

$$\frac{x^2}{\cos^2\theta} - 2xy\,\frac{\cos\delta}{\sin\theta\,\cos\theta} + \frac{y^2}{\sin^2\theta} = \sin^2\delta$$

The direction of the axes is obtained by looking for the eigenvectors of the matrix

$$A = \begin{pmatrix} \dfrac{1}{\cos^2\theta} & -\dfrac{\cos\delta}{\sin\theta\,\cos\theta} \\ -\dfrac{\cos\delta}{\sin\theta\,\cos\theta} & \dfrac{1}{\sin^2\theta} \end{pmatrix}$$

which make angles α and $\alpha + \pi/2$ with the x-axis, where α is given by

$$\tan\alpha = \cos\delta\tan 2\theta$$

The vector product of the position $\vec{r}$ with the velocity $\vec{v}$, $\vec{r}\times\vec{v}$, is easily seen to be

$$\vec{r}\times\vec{v} = \frac{1}{2}\,\omega\hat{z}\sin 2\theta\sin\delta$$

so that the sense of rotation is given by the sign of the product $\sin 2\theta\sin\delta$.

2. The intensity at the entrance of the polarizer is

$$I_0 = k\left(E_0^2\cos^2\theta + E_0^2\sin^2\theta\right) = kE_0^2$$

where k is a proportionality factor. At the exit of the polarizer it is

$$I = kE_0^2\cos^2\theta = I_0\cos^2\theta$$

The reduction of the intensity allows us to determine $|\cos\theta|$.

3. The projection of the electric field on the polarizer axis is

$$\frac{E_0}{\sqrt{2}}\left[\cos\theta\,\cos\omega t + \sin\theta\,\cos(\omega t - \delta)\right]$$

and the intensity is given by the time average

$$\begin{aligned} I' &= kE_0^2\Big\langle \cos^2\theta\,\cos^2\omega t + \sin^2\theta\,\cos^2(\omega t-\delta) + 2\sin\theta\cos\theta\,\cos\omega t\cos(\omega t-\delta)\Big\rangle \\ &= \frac{1}{2}\,kE_0^2(1+\sin 2\theta\cos\delta) = \frac{1}{2}\,I_0(1+\sin 2\theta\cos\delta) \end{aligned}$$

From the measurement of I' we deduce $\cos\delta$, which allows us to deduce δ up to a sign. The remaining ambiguities are lifted if one remarks that the ellipse is invariant under the transformations

$$\theta \to \theta + \pi \qquad \delta \to \delta$$

and

$$\theta \to -\theta \qquad \delta \to \delta + \pi$$

.

2.6.2 The (λ, μ) polarizer

1. The components $\mathcal{E}'_x$ and $\mathcal{E}'_y$ are given by

$$\begin{aligned}
\mathcal{E}'_x &= \mathcal{E}_x \cos^2\theta + \mathcal{E}_y \sin\theta\cos\theta\, \mathrm{e}^{-\mathrm{i}\eta} = |\lambda|^2 \mathcal{E}_x + \lambda\mu^* \mathcal{E}_y, \\
\mathcal{E}'_y &= \mathcal{E}_x \sin\theta\cos\theta\, \mathrm{e}^{\mathrm{i}\eta} + \mathcal{E}_y \sin^2\theta = \lambda^*\mu\, \mathcal{E}_x + |\mu|^2 \mathcal{E}_y.
\end{aligned}$$

2. This operation amounts to projection on $|\Phi\rangle$. In fact, if we choose to write the vectors $|x\rangle$ and $|y\rangle$ as column vectors

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then the projector $\mathcal{P}_\Phi$

$$\mathcal{P}_\Phi = |\Phi\rangle\langle\Phi| = \big(\lambda|x\rangle + \mu|y\rangle\big)\big(\lambda^*\langle x| + \mu^*\langle y|\big)$$

is represented by the matrix

$$\mathcal{P}_\Phi = \begin{pmatrix} |\lambda|^2 & \lambda\mu^* \\ \lambda^*\mu & |\mu|^2 \end{pmatrix}$$

3. Since $\mathcal{P}_\Phi = |\Phi\rangle\langle\Phi|$, we clearly have $\mathcal{P}_\Phi|\Phi\rangle = |\Phi\rangle$ and $\mathcal{P}_\Phi|\Phi_\perp\rangle = 0$, because

$$\langle\Phi|\Phi_\perp\rangle = -\lambda^*\mu^* + \mu^*\lambda^* = 0$$

2.6.3 Circular polarization and the rotation operator

1. In complex notation the fields $\mathcal{E}_x$ and $\mathcal{E}_y$ are written as

$$\mathcal{E}_x = \frac{1}{\sqrt{2}}\, E_0, \qquad \mathcal{E}_y = \frac{1}{\sqrt{2}}\, E_0\, \mathrm{e}^{\pm \mathrm{i}\pi/2} = \frac{\pm\mathrm{i}}{\sqrt{2}}\, E_0,$$

where the (+) sign corresponds to right-handed circular polarization and the (−) to left-handed. The proportionality factor E_0 common to $\mathcal{E}_x$ and $\mathcal{E}_y$ defines the intensity of the light wave and plays no role in describing the polarization, which is characterized by the normalized vectors

$$|\mathrm{R}\rangle = \frac{1}{\sqrt{2}}\,(|x\rangle + \mathrm{i}|y\rangle), \qquad |\mathrm{L}\rangle = \frac{1}{\sqrt{2}}\,(|x\rangle - \mathrm{i}|y\rangle)$$

2. Let us compute $|\mathrm{R}'\rangle$. We have

$$\begin{aligned}
|\mathrm{R}'\rangle &= \frac{1}{\sqrt{2}}\big(\cos\theta|x\rangle + \sin\theta|y\rangle - \mathrm{i}\sin\theta|x\rangle + \mathrm{i}\cos\theta|y\rangle\big) \\
&= \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\theta}\,|x\rangle + \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\,|y\rangle\right) = \mathrm{e}^{-\mathrm{i}\theta}\,|R\rangle
\end{aligned}$$

and similarly $|\mathrm{L}'\rangle = \exp(\mathrm{i}\theta)|\mathrm{L}\rangle$. The vectors $|\mathrm{R}'\rangle$ and $\mathrm{L}'\rangle$ differ from $|\mathrm{R}\rangle$ and $\mathrm{L}\rangle$ by a phase factor only, and they do not represent different physical states.

3. The projectors on the vectors $|\mathrm{R}\rangle$ and $|\mathrm{L}\rangle$ are given by

$$\mathcal{P}_\mathrm{D} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \qquad \mathcal{P}_\mathrm{G} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}$$

and Σ is

$$\Sigma = \mathcal{P}_{\rm D} - \mathcal{P}_{\rm G} = \begin{pmatrix} 0 & -{\rm i} \\ {\rm i} & 0 \end{pmatrix}$$

This operator has the states $|{\rm R}\rangle$ and $|{\rm L}\rangle$ as its eigenvectors, and their respective eigenvalues are $+1$ and -1:

$$\Sigma|{\rm R}\rangle = |{\rm R}\rangle, \qquad \Sigma|{\rm L}\rangle = -|{\rm L}\rangle$$

Thus $\exp(-{\rm i}\theta\Sigma)|{\rm R}\rangle = \exp(-{\rm i}\theta)|{\rm R}\rangle$ and $\exp(-{\rm i}\theta\Sigma)|{\rm L}\rangle = \exp({\rm i}\theta)|{\rm L}\rangle$

4. From the form of Σ in the $\{|x\rangle, |y\rangle\}$ basis we get at once $\Sigma^2 = I$, and thus

$${\rm e}^{-{\rm i}\theta\Sigma} = I - {\rm i}\theta\Sigma + \frac{(-{\rm i}\theta)^2}{2!} I + \frac{(-{\rm i}\theta)^3}{3!}\Sigma + \cdots$$

The series is easily summed with the result

$$\exp(-{\rm i}\theta\Sigma_z) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

If we apply the operator $\exp(-{\rm i}\theta\Sigma)$ to the vectors of the $\{|x\rangle, |y\rangle\}$ basis, we get the rotated vectors $|\theta\rangle$ and $|\theta_\perp\rangle$, so that this operator represents a rotation by an angle θ about the z axis.

2.6.4 An optimal strategy for Eve?

1. If Alice uses the $|x\rangle$ basis, the probability that Eve guesses correctly is $\mathsf{p}_x = \cos^2\phi$. If she uses the the $|\pm\pi/4\rangle$ basis, this probability is

$$\mathsf{p}_{\pi/4} = |\langle\phi|\pm\pi/4\rangle|^2 = \frac{1}{2}\,(\cos\phi + \sin\phi)^2$$

The probability that Eve guesses correctly is

$$\begin{aligned} \mathsf{p}(\phi) &= \frac{1}{2}\,(\mathsf{p}_x + \mathsf{p}_{\pi/4}) = \frac{1}{4}\left[2\cos^2\phi + (\cos\phi + \sin\phi)^2\right] \\ &= \frac{1}{4}\,[2 + \cos 2\phi + \sin 2\phi] \end{aligned}$$

The maximum of $\mathsf{p}(\phi)$ is given by $\phi = \phi_0 = \pi/8$, which is evident from symmetry considerations: the maximum must be given by the bisector of the Ox and $\pi/4$ axes. The maximum value is

$$\mathsf{p}_{\rm max} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \simeq 0.854$$

2. If Alice sends a $|\theta\rangle$ ($|\theta_\perp\rangle$) photon, Eve obtains the correct result with probability $\cos^2\theta$ ($\sin^2\theta$), and the probability that Bob receives the correct polarization is $\cos^4\theta$ ($\sin^4\theta$). The probability of success for Eve is

$$\mathsf{p}_{\rm s} = \frac{1}{2}\left(1 + \cos^4\theta + \sin^4\theta\right)$$

and the probability of error

$$\mathsf{p}_{\rm e} = 1 - \mathsf{p}_{\rm s} = \sin^2\theta\cos^2\theta = \frac{1}{4}\,\sin^2 2\theta$$

Eves's error are maximal for $\theta = \pi/4$.

Heisenberg inequalities

1. The commutator of A and B is of the form ${\rm i}C$, where C is a Hermitian operator because

$$[A, B]^\dagger = [B^\dagger, A^\dagger] = [B, A] = -[A, B].$$

We can then write

$$[A,B] = \mathrm{i}C, \qquad C = C^\dagger. \tag{2.1}$$

2. Let us define the Hermitian operators of zero expectation value (*a priori* specific to the state $|\varphi\rangle$):

$$A_0 = A - \langle A\rangle_\varphi I, \qquad B_0 = B - \langle B\rangle_\varphi I.$$

Their commutator is also $\mathrm{i}C$, $[A_0, B_0] = \mathrm{i}C$, because $\langle A\rangle_\varphi$ and $\langle B\rangle_\varphi$ are numbers. The squared norm of the vector

$$(A_0 + \mathrm{i}\lambda B_0)|\varphi\rangle,$$

where λ is chosen to be real, must be positive:

$$\begin{aligned} ||(A_0 + \mathrm{i}\lambda B_0)|\varphi\rangle||^2 &= ||A_0|\varphi\rangle||^2 + \mathrm{i}\lambda\langle\varphi|A_0B_0|\varphi\rangle - \mathrm{i}\lambda\langle\varphi|B_0A_0|\varphi\rangle + \lambda^2||B_0|\varphi\rangle||^2 \\ &= \langle A_0^2\rangle_\varphi - \lambda\langle C\rangle_\varphi + \lambda^2\langle B_0^2\rangle_\varphi \geq 0. \end{aligned}$$

The second-degree polynomial in λ must be positive for any λ, which implies

$$\langle C\rangle_\varphi^2 - 4\langle A_0^2\rangle_\varphi\langle B_0^2\rangle_\varphi \leq 0.$$

This demonstrates the Heisenberg inequality

$$\boxed{(\Delta_\varphi A)\,(\Delta_\varphi B) \geq \frac{1}{2}\left|\langle C\rangle_\varphi\right|}\,. \tag{2.2}$$

3. In a finite dimensional space, the trace of a commutator vanishes because $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$, so that the equality

$$[X, P] = \mathrm{i}\hbar I$$

cannot be realized in a finite dimensional space.

Chapter 3

Exercises from Chapter 3

3.5.1 Rotation operator for spin 1/2

1. We use $\sigma_x|0\rangle = |1\rangle$, $\sigma_x|1\rangle = |0\rangle$, $\sigma_y|0\rangle = \mathrm{i}|1\rangle$, $\sigma_y|1\rangle = -\mathrm{i}|1\rangle$, $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$ to obtain

$$\begin{aligned}
\sigma_x|\varphi\rangle &= \mathrm{e}^{\mathrm{i}\phi/2}\,\sin\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{-\mathrm{i}\phi/2}\,\cos\frac{\theta}{2}\,|1\rangle \\
\sigma_y|\varphi\rangle &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi/2}\,\sin\frac{\theta}{2}\,|0\rangle + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi/2}\,\cos\frac{\theta}{2}\,|1\rangle \\
\sigma_z|\varphi\rangle &= \mathrm{e}^{-\mathrm{i}\phi/2}\,\cos\frac{\theta}{2}\,|0\rangle - \mathrm{e}^{\mathrm{i}\phi/2}\,\sin\frac{\theta}{2}\,|1\rangle
\end{aligned}$$

so that

$$\begin{aligned}
\langle\varphi|\sigma_x|\varphi\rangle &= \sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\left(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}\right) = \sin\theta\cos\phi \\
\langle\varphi|\sigma_y|\varphi\rangle &= \sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\left(-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\right) = \sin\theta\sin\phi \\
\langle\varphi|\sigma_z|\varphi\rangle &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta
\end{aligned}$$

2. From (3.8) we derive the identity

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,I + \mathrm{i}\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

so that

$$(\vec{\sigma}\cdot\hat{p})^2 = I \qquad (\vec{\sigma}\cdot\hat{p})^3 = (\vec{\sigma}\cdot\hat{p})\cdots$$

and the series expansion of the exponential reads

$$\begin{aligned}
\exp\left(-\mathrm{i}\frac{\theta}{2}\,\vec{\sigma}\cdot\hat{p}\right) &= I + \left(\frac{-\mathrm{i}\theta}{2}\right)(\vec{\sigma}\cdot\hat{p}) + \frac{1}{2!}\left(\frac{-\mathrm{i}\theta}{2}\right)^2 I + \frac{1}{3!}\left(\frac{-\mathrm{i}\theta}{2}\right)^3(\vec{\sigma}\cdot\hat{p}) + \cdots \\
&= I\,\cos\frac{\theta}{2} - \mathrm{i}(\vec{\sigma}\cdot\hat{p})\,\sin\frac{\theta}{2}
\end{aligned}$$

The action of the operator $\exp(-\mathrm{i}\theta\,\vec{\sigma}\cdot\hat{p}/2)$ on the vector $|0\rangle$ is

$$\exp\left(-\mathrm{i}\frac{\theta}{2}\,\vec{\sigma}\cdot\hat{p}\right)|0\rangle = \cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\,\sin\frac{\theta}{2}\,|1\rangle$$

which is the same as (3.4) up to a physically irrelevant phase factor $\exp(\mathrm{i}\phi/2)$. Thus $\exp(-\mathrm{i}\theta\,\vec{\sigma}\cdot\hat{p}/2)$ is the operator which rotates the vector $|0\rangle$, the eigenvector of σ_z with eigenvalue 1, on $|\varphi\rangle$, the eigenvectors of $\vec{\sigma}\cdot\hat{n}$ with the same eigenvalue. The same result holds for the eigenvalue -1, corresponding to $|1\rangle$ and the rotated vector $U[\mathcal{R}_{\hat{p}}(\theta)]|1\rangle$.

3. Let us specialize the above results to the case $\phi = -\pi/2$, which corresponds to rotations about the x-axis

$$U[\mathcal{R}_x(\theta)] = \exp\left(-\mathrm{i}\frac{\theta}{2}\,\sigma_x\right) = \begin{pmatrix} \cos(\theta/2) & -\mathrm{i}\sin(\theta/2) \\ -\mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

A rotation about the x-axis transforms $|0\rangle$ into the vector

$$|\varphi\rangle = \cos\frac{\theta}{2}\,|0\rangle - \mathrm{i}\sin\frac{\theta}{2}\,|1\rangle$$

Taking $\theta = -\omega_1 t$, this corresponds exactly to (3.31) with the initial conditions $a = 1, b = 0$.

3.5.2 Rabi oscillations away from resonance

1. Substituting in the differential equation the exponential form of $\hat{\lambda}(t)$, we get the second order equation for $\Omega_\pm$

$$2\Omega_\pm^2 - 2\delta\Omega_\pm - \frac{1}{2}\,\omega_1^2 = 0$$

whose solutions are

$$\Omega_\pm = \frac{1}{2}\left[\delta \pm \sqrt{\delta^2 + \omega_1^2}\right] = \frac{1}{2}\,[\delta \pm \Omega]$$

2. The solution of the differential equation for $\hat{\lambda}$ is a linear combination of $\exp(\mathrm{i}\Omega_+ t)$ and $\exp(\mathrm{i}\Omega_- t)$

$$\hat{\lambda}(t) = a\,\exp(\mathrm{i}\Omega_+ t) + b\,\exp(\mathrm{i}\Omega_- t).$$

Let us choose the initial conditions $\hat{\lambda}(0) = 1$, $\hat{\mu}(0) = 0$. Since $\hat{\mu}(0) \propto \mathrm{d}\hat{\lambda}/\mathrm{d}t(0)$, these initial conditions are equivalent to

$$a + b = 1 \quad \text{and} \quad a\Omega_+ - b\Omega_- = 0,$$

and so

$$a = -\frac{\Omega_-}{\Omega}, \qquad b = \frac{\Omega_+}{\Omega}$$

The final result can be written as

$$\begin{aligned} \hat{\lambda}(t) &= \frac{\mathrm{e}^{\,\mathrm{i}\delta t/2}}{\Omega}\left[\Omega\cos\frac{\Omega t}{2} - \mathrm{i}\delta\sin\frac{\Omega t}{2}\right], \\ \hat{\mu}(t) &= \frac{\mathrm{i}\omega_1}{\Omega}\mathrm{e}^{\,-\mathrm{i}\delta t/2}\sin\frac{\Omega t}{2}, \end{aligned}$$

which reduces to (3.31) when $\delta = 0$. The factor $\exp(\pm\mathrm{i}\delta t/2)$ arises because δ is the Larmor frequency in the rotating reference frame. The second equation shows that if we start from the state $|0\rangle$ at $t = 0$, the probability of finding the spin in the state $|1\rangle$ at time t is

$$\mathsf{p}_{0\to 1}(t) = \frac{\omega_1^2}{\Omega^2}\,\sin^2\left(\frac{\Omega t}{2}\right)$$

We see that the maximum probability of making a transition from the state $|0\rangle$ to the state $|1\rangle$ for $\Omega t/2 = \pi/2$ is given by a resonance curve of width δ:

$$\mathsf{p}_-^{\max} = \frac{\omega_1^2}{\Omega^2} = \frac{\omega_1^2}{\omega_1^2 + \delta^2} = \frac{\omega_1^2}{\omega_1^2 + (\omega - \omega_0)^2}$$

Chapter 4

Exercises from Chapter 4

Basis independence of the tensor product

The tensor product $|i_A \otimes j_B\rangle$ is given by

$$|i_A \otimes j_B\rangle = \sum_{m,n} R_{im} S_{jn} |m_A \otimes n_B\rangle$$

Let us define $|\varphi_A \otimes \chi_B\rangle'$ by using the $\{|i_A\rangle, |j_B\rangle\}$ bases

$$\begin{aligned} |\varphi_A \otimes \chi_B\rangle' &= \sum_{i,j} \hat{c}_i \hat{d}_j |i_A \otimes j_B\rangle \\ &= \sum_{i,j,m,n} \hat{c}_i \hat{d}_j R_{im} S_{in} |m_A \otimes n_B\rangle \end{aligned}$$

We can now use the transformation law of the components in a change of basis

$$\hat{c}_i = \sum_k R^{-1}_{ki} c_k \qquad \hat{d}_j = \sum_l S^{-1}_{lj} d_l$$

to show that

$$|\varphi_A \otimes \chi_B\rangle' = \sum_{m,n} c_m d_n |m_A \otimes n_B\rangle = |\varphi_A \otimes \chi_B\rangle$$

Thus the tensor product is independent of the choice of basis.

4.6.2 Properties of the state operator

1. Since the p_i are real, ρ is clearly Hermitian. Furthermore $\mathrm{Tr}\,\rho = \sum_i \mathsf{p}_i = 1$, and finally ρ is positive as

$$\langle\varphi|\rho|\varphi\rangle = \sum_i \mathsf{p}_i |\langle\varphi|i\rangle|^2 \geq 0$$

Let us first compute $\mathrm{Tr}\,(M|i\rangle\langle i|)$

$$\mathrm{Tr}\,(M|i\rangle\langle i|) = \sum_j \langle j|M|i\rangle\langle i|j\rangle = \langle i|M|i\rangle$$

whence

$$\mathrm{Tr}\left(\sum_i \mathsf{p}_i M|i\rangle\langle i|\right) = \sum_i \mathsf{p}_i \langle i|M|i\rangle$$

The expectation value of M in the state $|i\rangle$ appears in the sum over i with the weight p_i, as expected.

2. In the $|i\rangle$ basis, ρ has a diagonal form with matrix elements $\rho_{ii} = \mathsf{p}_i$, so that $\rho^2 = \rho$ can only hold if one of the probabilities is one, as the equation $\mathsf{p}_i^2 = \mathsf{p}_i$ has solutions $\mathsf{p}_i = 1$ and $\mathsf{p}_i = 0$. Furthermore, $\mathrm{Tr}\,\rho^2 = \sum_i \mathsf{p}_i^2$ and $\sum_i \mathsf{p}_i^2 \leq \sum_i \mathsf{p}_i$, where the equality holds if and only if one of the p_i is equal to one. Let us assume, for example, that $\mathsf{p}_1 = 1$, $\mathsf{p}_i = 0, i \neq 1$. Then $\rho = |1\rangle\langle 1|$, which corresponds to the pure state $|1\rangle$. One can also remark that $\rho^2 = \rho$ implies that ρ is a projector $\mathcal{P}$, and the rank of this projector is one, because $\mathrm{Tr}\,\mathcal{P}$ is the dimension of the subspace on which $\mathcal{P}$ projects.

4.6.3 The state operator for a qubit and the Bloch vector

The condition for a Hermitian 2×2 matrix is $\rho_{01} = \rho_{10}^*$, so that

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix}$$

is indeed the most general 2×2 Hermitian matrix with trace one. The eigenvalues λ_+ and λ_- of ρ satisfy

$$\lambda_+ + \lambda_- = 1, \qquad \lambda_+\lambda_- = \det\rho = a(1-a) - |c|^2,$$

and we must have $\lambda_+ \geq 0$ and $\lambda_- \geq 0$. The condition $\det\rho \geq 0$ implies that λ_+ and λ_- have the same sign, and the condition $\lambda_+ + \lambda_- = 1$ implies that $\lambda_+\lambda_-$ reaches its maximum for $\lambda_+\lambda_- = 1/4$, so that finally

$$0 \leq a(1-a) - |c|^2 \leq \frac{1}{4}$$

The necessary and sufficient condition for ρ to describe a pure state is

$$\det\rho = a(1-a) - |c|^2 = 0.$$

The coefficients a and c for the state matrix describing the normalized state vector $|\psi\rangle = \lambda|+\rangle + \mu|-\rangle$ with $|\lambda|^2 + |\mu|^2 = 1$ are

$$a = |\lambda|^2 \qquad c = \lambda\mu^*$$

so that $a(1-a) = |c|^2$ in this case.

2. Since any 2×2 Hermitian matrix can be written as a linear combination of the unit matrix I and the σ_i with reals coefficients, we can write the state matrix as

$$\rho = \frac{I}{2} + \sum_i b_i\sigma_i = \frac{1}{2}\left(I + \vec{b}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1+b_z & b_x - \mathrm{i}b_y \\ b_x + \mathrm{i}b_y & 1-b_z \end{pmatrix}$$

where we have used $\mathrm{Tr}\sigma_i = 0$. The vector $\vec{b}$, called the *Bloch vector*, must satisfy $|\vec{b}|^2 \leq 1$ owing to the results of question **1**, and a pure state corrresponds to $|\vec{b}|^2 = 1$. Let us calculate the expectation value of $\vec{\sigma}$ using $\mathrm{Tr}\,\sigma_i\sigma_j = 2\delta_{ij}$. We find

$$\langle\sigma_i\rangle = \mathrm{Tr}\,(\rho\,\sigma_i) = b_i$$

so that $\vec{b}$ is the expectation value $\langle\vec{\sigma}\rangle$.

3. With $\vec{B}$ parallel to Oz, the Hamiltonian reads

$$H = -\frac{1}{2}\gamma\sigma_z$$

The evolution equation

$$\mathrm{i}\hbar\frac{\mathrm{d}|\varphi(t)\rangle}{\mathrm{d}t} = H|\varphi\rangle$$

translates into the following for the state matrix

$$\mathrm{i}\hbar\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = [H, \rho]$$

so that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[H, \rho] = -\frac{1}{2}\gamma B(b_x\sigma_y - b_y\sigma_x)$$

which is equivalent to

$$\frac{\mathrm{d}b_x}{\mathrm{d}t} = -\gamma B b_y \qquad \frac{\mathrm{d}b_y}{\mathrm{d}t} = \gamma B b_x \qquad \frac{\mathrm{d}b_x}{\mathrm{d}t} = 0$$

This can be put in vector form

$$\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} = -\gamma \vec{B} \times \vec{b}$$

This equation shows that the Bloch vector rotates about the Oz axis with an angular frequency $\omega = \gamma B$.

4.6.4 The SWAP operator

1. Let us write explicitly the action of σ_x et σ_y on the vectors $|\varepsilon_1\varepsilon_2\rangle$

$$\begin{aligned}
\sigma_{1x}\sigma_{2x}|++\rangle &= |--\rangle \qquad & \sigma_{1y}\sigma_{2y}|++\rangle = -|--\rangle \\
\sigma_{1x}\sigma_{2x}|+-\rangle &= |-+\rangle \qquad & \sigma_{1y}\sigma_{2y}|+-\rangle = |-+\rangle \\
\sigma_{1x}\sigma_{2x}|-+\rangle &= |+-\rangle \qquad & \sigma_{1y}\sigma_{2y}|-+\rangle = |+-\rangle \\
\sigma_{1x}\sigma_{2x}|--\rangle &= |++\rangle \qquad & \sigma_{1y}\sigma_{2y}|--\rangle = -|++\rangle
\end{aligned}$$

Furthermore, $\sigma_{1z}\sigma_{2z}|\varepsilon_1\varepsilon_2\rangle = \varepsilon_1\varepsilon_2|\varepsilon_1\varepsilon_2\rangle$, whence the action of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ on the basis vectors

$$\begin{aligned}
\vec{\sigma}_1 \cdot \vec{\sigma}_2|++\rangle &= |++\rangle \\
\vec{\sigma}_1 \cdot \vec{\sigma}_2|+-\rangle &= 2|-+\rangle - |+-\rangle \\
\vec{\sigma}_1 \cdot \vec{\sigma}_2|-+\rangle &= 2|+-\rangle - |-+\rangle \\
\vec{\sigma}_1 \cdot \vec{\sigma}_2|--\rangle &= |--\rangle
\end{aligned}$$

Then one obtains immediately

$$\frac{1}{2}(I + \vec{\sigma}_A \cdot \vec{\sigma}_B)|i_A j_B\rangle = |j_A i_B\rangle$$

4.6.5 The Schmidt purification theorem

Let us choose as a basis of $\mathcal{H}_A$ a set $\{|m_A\rangle\}$ which diagonalizes the reduced state operator ρ_A:

$$\rho_A = \mathrm{Tr}_B|\varphi_{AB}\rangle\langle\varphi_{AB}| = \sum_{m=1}^{N_S} \mathsf{p}_m|m_A\rangle\langle m_A|$$

If the number N_S of nonzero coefficients p_m is smaller than the dimension N_A of $\mathcal{H}_A$, we complete the set $\{|m_A\rangle\}$ by a set of $(N_A - N_S)$ orthonormal vectors, chosen to be orthogonal to the space spanned by the vectors $|m_A\rangle$. We use (4.12) to compute ρ_A from $|\varphi_{AB}\rangle$

$$\rho_A = \sum_{m,n} \langle\tilde{n}_B|\tilde{m}_B\rangle|m_A\rangle\langle n_A|$$

On comparing the two expressions of ρ_A we see that

$$\langle\tilde{n}_B|\tilde{m}_B\rangle = \mathsf{p}_m\delta_{mn},$$

and with our choice of basis $\{|m_A\rangle\}$ it turns out that the vectors $\{|\tilde{m}_B\rangle\}$ are, after all, orthogonal. To obtain an orthonormal basis, we only need to rescale the vectors $|\tilde{n}_B\rangle$

$$|n_B\rangle = \mathsf{p}_n^{-1/2}|\tilde{n}_B\rangle,$$

where we may assume that $\mathsf{p}_n > 0$ because, as explained above, it is always possible to complete the basis of $\mathcal{H}_B$ by a set of $(N_B - N_S)$ orthonormal vectors. We finally obtain Schmidt's decomposition of $|\varphi_{AB}\rangle$ on an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\varphi_{AB}\rangle = \sum_n \mathsf{p}_n^{1/2} |n_A \otimes n_B\rangle.$$

Any pure state $|\varphi_{AB}\rangle$ may be written in the preceding form, but the bases $\{|n_A\rangle\}$ and $\{|n_B\rangle\}$ will of course depend on the state under consideration. If some of the p_n are equal, then the decomposition is not unique, as is the case for the spectral decomposition of a Hermitian operator with degenerate eigenvalues. The reduced state operator ρ_B is readily computed from (4.12) using the orthogonality condition $\langle m_A | n_A \rangle = \delta_{mn}$:

$$\rho_B = \mathrm{Tr}_A |\varphi_{AB}\rangle\langle\varphi_{AB}| = \sum_n \mathsf{p}_n |n_B\rangle\langle n_B|$$

4.6.6 A model for phase damping

The state matrix at time t is

$$\rho(t) = \begin{pmatrix} \langle|\lambda(t)|^2\rangle & \langle\lambda(t)\mu^*(t)\rangle \\ \langle\lambda^*(t)\mu(t)\rangle & \langle|\mu(t)|^2\rangle \end{pmatrix}$$

where $\langle\bullet\rangle$ stands for an average over all the realizations of the random function. Clearly $\langle|\lambda(t)|^2\rangle$ and $\langle|\mu(t)|^2\rangle$ are time-independent, so that the populations are time-independent. However, the coherences depend on time. Let us compute the average of $\lambda(t)\mu^*(t)$

$$\begin{aligned} \langle\lambda(t)\mu^*(t)\rangle &= \lambda_0\mu_0^* \Big\langle \exp\left(\mathrm{i}\int_0^t \omega(t')\mathrm{d}t'\right) \Big\rangle \\ &= \lambda_0\mu_0^* \exp\left(\mathrm{i}\,\langle\omega_0\rangle t\right) \exp\left(-\frac{1}{2}\int_0^t C(t-t')\mathrm{d}t'\mathrm{d}t''\right) \end{aligned}$$

where we have used a standard property of Gaussian random functions. We thus obtain

$$\rho_{01}(t) = \rho_{01}(t=0) \exp\left(\mathrm{i}\,\langle\omega_0\rangle t\right) \exp\left(-\frac{1}{2}\int_0^t C(t-t')\mathrm{d}t'\mathrm{d}t''\right)$$

If we assume that $t \gg \tau$, then

$$\int_0^t \mathrm{d}t'\mathrm{d}t''\mathrm{e}^{-|t'-t''|/\tau} \simeq 2t\int_0^\infty \mathrm{d}t\mathrm{e}^{-t/\tau} = 2t\tau$$

and

$$\rho_{01}(t) = \rho_{01}(t=0)\mathrm{e}^{\mathrm{i}\langle\omega_0\rangle t}\,\mathrm{e}^{-C\tau t}$$

4.6.7 Amplitude damping channel

1 The evolution of $|\Phi\rangle$ during Δt is

$$|\Phi\rangle \to U|\Phi\rangle = \lambda|0_A \otimes 0_E\rangle + \mu\sqrt{1-\mathsf{p}}\,|1_A \otimes 0_E\rangle + \mu\sqrt{\mathsf{p}}\,|1_A \otimes 1_E\rangle$$

In order to obtain the state matrix of system A, we take the trace over the environment

$$\begin{aligned} \mathrm{Tr}_E(U|\Phi\rangle\langle\Phi|U^\dagger) &= (|\lambda|^2 + \mathsf{p}|\mu|^2)|0_A\rangle\langle 0_A| + \lambda\mu^*\sqrt{1-\mathsf{p}}\,|0_A\rangle\langle 1_A| \\ &+ \lambda^*\mu\sqrt{1-\mathsf{p}}\,|1_A\rangle\langle 0_A| + (1-\mathsf{p})|\mu|^2|1_A\rangle\langle 1_A| \end{aligned}$$

or, in matrix from

$$\rho^{(1)} = \rho(\Delta t) = \begin{pmatrix} 1-(1-\mathsf{p})|\mu|^2 & \sqrt{1-\mathsf{p}}\,\lambda\mu^* \\ \sqrt{1-\mathsf{p}}\,\lambda^*\mu & (1-\mathsf{p})|\mu|^2 \end{pmatrix}$$

After n iterations we get

$$\rho^{(n)} = \rho(n\Delta t) = \begin{pmatrix} 1-(1-\mathsf{p})^n|\mu|^2 & (1-\mathsf{p})^{n/2}\,\lambda\mu^* \\ (1-\mathsf{p})^{n/2}\,\lambda^*\mu & (1-\mathsf{p})^n|\mu|^2 \end{pmatrix}$$

Using in the limit $\Delta t \to 0$ the relation

$$\lim_{\Delta t\to 0}(1-\Gamma\Delta t)^{t/\Delta t} = \mathrm{e}^{-\Gamma t}$$

we get the expression given in the statement of the problem. We clearly have $T_1 = 1/\Gamma$ and $T_2 = 2/\Gamma$, so that $T_2 = 2T_1$.

2. If we detect no photons, we know that we have prepared the atom in the (unnormalized) state

$$\lambda|0_A\rangle + \mu\sqrt{1-\mathsf{p}}\,|1_A\rangle$$

The failure to detect a photon has changed the state of the atom!

4.6.8 Invariance of the Bell states under rotation

We have

$$\begin{aligned} |x_A x_B\rangle &= (\cos\theta|\theta_A\rangle - \sin\theta|\theta_{A\perp}\rangle)(\cos\theta|\theta_B\rangle - \sin\theta|\theta_{B\perp}\rangle) \\ |y_A x_B\rangle &= (\sin\theta|\theta_B\rangle + \cos\theta|\theta_{B\perp}\rangle)(\sin\theta|\theta_B\rangle + \cos\theta|\theta_{B\perp}\rangle) \end{aligned}$$

and an explicit calculation immediately gives

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|x_A x_B\rangle + |y_A y_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|\theta_A\theta_B\rangle + |\theta_{A\perp}\theta_{B\perp}\rangle\right)$$

Chapter 5

Exercises from chapter 5

5.10.1 Justification of the figures of Fig. 5.4

1. The upper circuit of Fig. 5.4 reads in matrix form

$$\begin{aligned} M &= \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \\ &= \begin{pmatrix} CBA & 0 \\ 0 & C\sigma_x B\sigma_x A \end{pmatrix} \end{aligned}$$

where the matrices have been written in block diagonal form with 2×2 matrices. Then we must find three matrices, A, B and C such that

$$CBA = I \qquad C\sigma_x B\sigma_x A = U$$

Action of the cNOT gate

$$\text{cNOT}\left[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right] = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This is an entangled state (in fact it is one of the four Bell states).

2. Let us define the unitary matrix U by

$$U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

with

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1 \qquad \alpha\beta^* + \gamma\delta^* = \alpha\gamma^* + \beta\delta^* = 0$$

and start from the most general two-qubit state

$$|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

Assume we measure the control bit and find it in the $|0\rangle$ state. Then the state vector of the target bit is

$$|\varphi_0\rangle = a|0\rangle + b|1\rangle$$

If we find the control bit in state $|1\rangle$, then we apply U to the state $|\varphi\rangle = c|0\rangle + d|1\rangle$

$$U|\varphi\rangle = |\varphi_1\rangle = (\alpha c + \gamma d)|0\rangle + (\beta c + \delta d)|1\rangle$$

On the other hand, if we apply the cU gate to $|\Psi\rangle$, the result is

$$\begin{aligned} \text{cU}|\Psi\rangle &= a|00\rangle + b|01\rangle + (\alpha c + \gamma d)|10\rangle + (\beta c + \delta d)|11\rangle \\ &= |0 \otimes \varphi_0\rangle + |1 \otimes \varphi_1\rangle \end{aligned}$$

3. It is clear that the vectors $|000\rangle$ and $|001\rangle$ are not modified by the lower circuit on the left of Fig. 5.4, so that we may start from the vector

$$|\Psi\rangle = a|010\rangle + b|011\rangle + c|100\rangle + d|101\rangle + e|110\rangle_f|111\rangle$$

We apply on $|\Psi\rangle$ the first gate on the left, the $\mathrm{c_2U_3}$ gate (with obvious notations)

$$\begin{aligned} \mathrm{c_2U_3}|\Psi\rangle &= |\Psi_1\rangle = (\alpha a + \gamma b)|010\rangle + (\beta a + \delta b)|011\rangle + c|100\rangle \\ &+ d|101\rangle + (\alpha e + \gamma f)|110\rangle + (\beta e + \delta f)|111\rangle \end{aligned}$$

where the matrix U is defined in the preceding question. The transformation law is thus

$$\begin{array}{lccl} a \to \alpha a + \gamma b & b & \to & \beta a + \delta b \qquad c \to c \\ d \to d & e & \to & \alpha e + \gamma f \qquad f \to \beta e + \delta f \end{array}$$

Let us give as an intermediate result of the calculation

$$|\Psi_4\rangle = (\mathrm{c_1NOT_2})(\mathrm{c_2U_3})(\mathrm{c_1NOT_2})(\mathrm{c_2U_3})|\Psi\rangle$$

We find

$$\begin{aligned} |\Psi_4\rangle &= a|010\rangle + b|011\rangle + (\alpha^* c + \beta^* d)|100\rangle \\ &+ (\gamma^* c + \delta^* d)|101\rangle + (\alpha e + \gamma f)|110\rangle + (\beta e + \delta f)|111\rangle \end{aligned}$$

Finally

$$\begin{aligned} |\Psi_5\rangle &= \mathrm{c_1U_3}|\Psi_4\rangle = a|010\rangle + b|011\rangle + c|100\rangle + d|101\rangle \\ &+ (U^2_{11}e + U^2_{12}f)|110\rangle + (U^2_{21}e + U^2_{22}f)|111\rangle \end{aligned}$$

where U^2_{ij} is a matrix element of of the matrix U^2, for example

$$U^2_{11} = \alpha^2 + \beta\gamma$$

This gives preciseley the action of the Toffoli gate T_{U^2}. A non trivial action is obtained only if both control bits 1 and 2 are in the $|1\rangle$ state

$$T_{U^2}(|110\rangle + |111\rangle) = (U^2_{11}e + U^2_{12}f)|110\rangle + (U^2_{21}e + U^2_{22}f)|111\rangle$$

5.10.2 The Deutsch-Jozsa algorithm

1. Before entering the box U_f, the two upper qubits are in the state

$$\begin{aligned} H^{\otimes 2}|00\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}\sum_{x=0}^{3}|x\rangle \end{aligned}$$

2. From the results of Sec. 5.5

$$U_f|\Psi\rangle = \left(\frac{1}{2}\sum_{x=0}^{3}(-1)^{f(x)}|x\rangle\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

so that, calling $|\Psi'\rangle$ the state of the two upper qubits

(i) $f(x) = \mathrm{cst}$

$$|\Psi'\rangle = \pm\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

(i) $f(x) = x \bmod 2$

$$|\Psi'\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

3. Since $H^2 = I$, in case (i) we get

$$H^{\otimes 2}|\Psi'\rangle = \pm|00\rangle$$

while in case (ii) we may write

$$|\Psi'\rangle = \frac{1}{2}(|0\rangle \otimes (|0\rangle - |1\rangle) + (|0\rangle - |1\rangle) \otimes |1\rangle)$$

and

$$H^{\otimes 2}|\Psi'\rangle = H^{\otimes 2}\frac{1}{2}\left[(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)\right] = |01\rangle$$

The first qubit is in the state $|0\rangle$ and the second in the state $|1\rangle$. Note that the result of the final measurement of the upper qubits is unambiguous only if $|\Psi'\rangle$ is a non entangled state, so that we must have

$$(-1)^{f(0)+f(3)} = (-1)^{f(1)+f(2)}$$

5.10.3 Grover algorithm and constructive interference

Let us first apply the oracle O on $|\Psi\rangle$

$$O|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{f(x)}\,|x\rangle = \frac{1}{\sqrt{N}}\sum_x a_x|x\rangle$$

Then we apply $G = 2|\Psi\rangle\langle\Psi| - I$ using $\langle\Psi|x\rangle = 1/\sqrt{N}$

$$\begin{aligned} GO|\Psi\rangle &= \left(\frac{2}{N}\sum_y a_y\right)|\Psi\rangle - \frac{1}{\sqrt{N}}\sum_x a_x|x\rangle \\ &= \frac{1}{\sqrt{N}}\sum_x\left[\frac{2}{N}\sum_y a_y - a_x\right]|x\rangle = \frac{1}{\sqrt{N}}\sum_x a_x^{(1)}|x\rangle \end{aligned}$$

This gives us the relation, with $a_x^{(0)} = 1$

$$a_x^{(1)} = \frac{2}{N}\left(\sum_y(-1)^{f(y)}a_y^0\right) - (-1)^{f(x)}a_x^{(0)}$$

which leads to the recursion relation

$$a_x^{(n+1)} = \frac{2}{N}\left(\sum_y(-1)^{f(y)}a_y^n\right) - (-1)^{f(x)}a_x^{(n)}$$

If, for example, $N = 16$, then

(i) For $x_i \neq x_0$, $a_i^{(1)} = \frac{3}{4}$

(ii) For $x_i = x_0$, $a_i^{(1)} = \frac{11}{4}$

The probability (ii) for finding x_0 is greater than the probability (i) for finding x_i by a factor $121/9 \simeq 13.4$. A good check of the calculation is that the final sate vector is normalized to one: $15(3/4)^2 + (11/4)^2 = 1$!

5.10.4 Example of finding y_j

The probability for finding y_j is given by (5.45) with, in our specific case, $K = 5$, $n = 4$ and $r = 3$

$$\mathsf{p}(y_j) = \frac{1}{2^n K} \frac{\sin^2(\pi\delta_j K r/2^n)}{\sin^2(\pi\delta_j r/2^n)} = \frac{1}{80} \frac{\sin^2(15\pi\delta_j/16)}{\sin^2(3\pi\delta_j/16)}$$

The possible values of j are $j = 0$, $j = 1$, $j = 2$ and $j = 3$. To the first value corresponds $y_j = 0$ and $\delta_j = 0$. To the second one corresponds $y_j = 5$ with $|\delta_j| = .33$ and to the third one $y_j = 11$ with $|\delta_j| = .33$, while there is no y_j with $|\delta_j| < 1/2$ for the last one. We obtain for the probabilities

$$\mathsf{p}(0) = \frac{5}{16} \qquad \mathsf{p}(1) = \mathsf{p}(2) = .225$$

so that

$$\mathsf{p}(0) + \mathsf{p}(1) + \mathsf{p}(2) = 0.76 > 0.4$$

Assume, for example, that the measurement of the final qubits gives $y_j = 11$. Then we deduce that $j = 2$ and $r = 3$.

Chapter 6

Exercises from chapter 6

6.5.1 Off-resonance Rabi oscillations

From exercise 3.5.1, we kow that $\exp(-\mathrm{i}\theta(\vec{\sigma}\cdot\hat{p})/2)$ is the rotation operator by θ of a spin 1/2 about an axis $\hat{p}$. The vector $\hat{n}$ being normalized ($\hat{n}^2 = 1$), we have

$$\exp(-\mathrm{i}\tilde{H}t/\hbar) = I\cos\frac{\Omega t}{2} - \mathrm{i}(\vec{\sigma}\cdot\hat{n})\sin\frac{\Omega t}{2}$$

with

$$\vec{\sigma}\cdot\hat{n} = -\frac{\omega_1}{\Omega}\,\sigma_x + \frac{\delta}{\Omega}\,\sigma_z$$

so that the matrix form of $\exp(-\mathrm{i}\tilde{H}t/\hbar)$ is

$$\mathrm{e}^{-\mathrm{i}\tilde{H}t/\hbar} = \begin{pmatrix} \cos\frac{\Omega t}{2} + \frac{\delta}{\Omega}\sin\frac{\Omega t}{2} & \mathrm{i}\,\frac{\omega_1}{\Omega}\,\sin\frac{\Omega t}{2} \\ \mathrm{i}\,\frac{\omega_1}{\Omega}\,\sin\frac{\Omega t}{2} & \cos\frac{\Omega t}{2} - \frac{\delta}{\Omega}\sin\frac{\Omega t}{2} \end{pmatrix}$$

6.5.2 Commutation relations between the a and $a^\dagger$

1. The commutator of a and $a^\dagger$ is, from the definition (6.26)

$$\begin{aligned} [a, a^\dagger] &= \frac{M\omega_z}{2\hbar}\left[z + \frac{\mathrm{i}p_z}{M\omega_z}, z - \frac{\mathrm{i}p_z}{M\omega_z}\right] \\ &= \frac{M\omega_z}{2\hbar}\,\frac{-2\mathrm{i}}{M\omega_z}\,[z, p_z] = I \end{aligned}$$

2. To compute the commutator $[a^\dagger, a]$, we use the identity

$$[AB, C] = A[B, C] + [A, C]B$$

and we find

$$[a^\dagger a, a] = a^\dagger[a, a] + [a^\dagger, a]a = -a$$

6.5.2 Quantum computing with trapped ions

1. We write the interaction Hamiltonian in terms of σ_+ and σ_-

$$H_{\text{int}} = -\frac{1}{2}\,\hbar\omega_1[\sigma_+ + \sigma_-]\left[\mathrm{e}^{\mathrm{i}(\omega t - kz - \phi)} + \mathrm{e}^{-\mathrm{i}(\omega t - kz - \phi)}\right]$$

and go to the interaction picture using (6.5)

$$e^{iH_0t/\hbar}\,\sigma_\pm\,e^{-iH_0t/\hbar} = e^{\mp i\omega_0 t}\,\sigma_\pm$$

In the rotating wave approximation, we can neglect terms which behave as $\exp[\pm i(\omega_0+\omega)t]$ and we are left with

$$\tilde{H}_{\rm int} \simeq -\frac{\hbar}{2}\,\omega_1\Big[\sigma_+\,e^{i(\delta t-\phi)}\,e^{-ik\tilde{z}} + \sigma_-\,e^{-i(\delta t-\phi)}\,e^{ik\tilde{z}}\Big]$$

2. $\Delta z = \sqrt{\hbar/(2M\omega_z)}$ is the spread of the wave function in the harmonic well. Thus, $\eta = k\Delta z$ is the ratio of this spread to the wavelength of the laser light. We may write

$$k\tilde{z} = k\sqrt{\frac{\hbar}{2M\omega_z}}\,(a+a^\dagger) = \eta(a+a^\dagger)$$

The matrix element of $\tilde{H}_{\rm int}$ between the states $|1,m+m'\rangle$ and $|0,m\rangle$ is

$$\langle 1,m+m'|\tilde{H}_{\rm int}|m\rangle = -\frac{1}{2}\,\hbar\omega_1 e^{i(\delta t-\phi)}\,\langle m+m'|e^{-i\eta(a+a^\dagger)}|m\rangle$$

The Rabi frequency for oscillations between the two levels is

$$\omega_1^{m\to m+m'} = \omega_1|\langle m+m'|e^{-i\eta(a+a^\dagger)}|m\rangle|$$

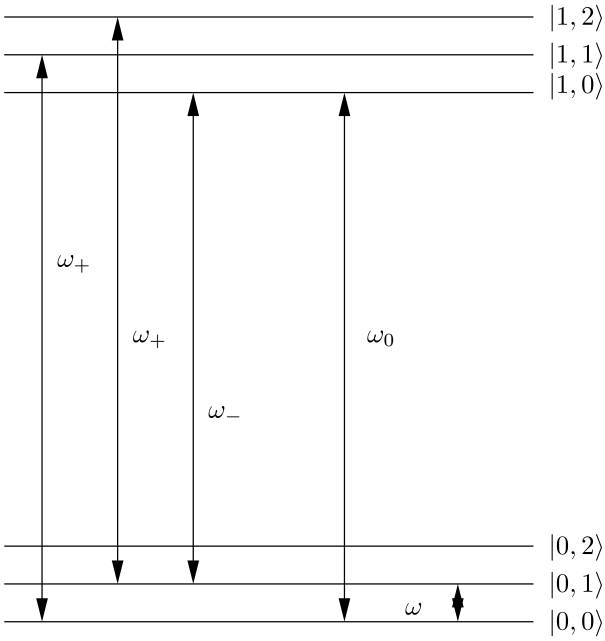

Figure 6.1: The level scheme. The transitions which are used are $(0,0)\leftrightarrow(0,1)$ and $(0,1)\leftrightarrow(1.2)$: bluesideband, $\omega_+=\omega_0+\omega_z$ and $(0,1)\leftrightarrow(1,1)$: red sideband, $\omega_-=\omega_0-\omega_z$.

3. Writing

$$e^{\pm i\eta(a+a^\dagger)} \simeq I \pm i\eta(a+a^\dagger)$$

and keeping terms to first order in η we get

$$\begin{aligned}\tilde{H}_{\rm int} &= \frac{i}{2}\eta\hbar\omega_1\Big[\sigma_+\;a\,e^{i(\delta-\omega_z)t}\,e^{-i\phi} - \sigma_-\;a^\dagger e^{-i(\delta-\omega_z)t}\,e^{i\phi}\\ &+\;\sigma_+\,a^\dagger e^{i(\delta+\omega_z)t}\,e^{-i\phi} - \sigma_-\;a\,e^{-i(\delta+\omega_z)t}\,e^{i\phi}\Big]\end{aligned}$$

The first line of $\tilde{H}_{\rm int}$ corresponds to a resonance at $\delta = \omega - \omega_0 = \omega_z$, that is, $\omega = \omega_0 + \omega_z$, a blue side band, and the second line to a resonance at $\omega = \omega_0 - \omega_z$, that is, a red sideband. The $\sigma_+ a$ term of the blue sideband induces transitions from $|0, m+1\rangle$ to $|1, m\rangle$, and the $\sigma_- a^\dagger$ term from $|1, m\rangle$ to $|0, m+1\rangle$. Now

$$\langle m|a|m+1\rangle = \langle m+1|a^\dagger|m\rangle = \sqrt{m+1}$$

so that we get $\tilde{H}^+_{\rm int}$ as written in the statement of the problem with

$$a_b = \frac{a}{\sqrt{m+1}} \qquad a_b^\dagger = \frac{a^\dagger}{\sqrt{m+1}}$$

The Rabi frequency is then $\omega_1\sqrt{m+1}$. The same reasoning may be applied to the red sideband.

4. The rotation operators $R(\theta, \phi)$ are given by

$$\begin{aligned} R(\theta, \phi = 0) &= I\cos\frac{\theta}{2} - {\rm i}\sigma_x \sin\frac{\theta}{2} \\ R\left(\theta, \phi = \frac{\pi}{2}\right) &= I\cos\frac{\theta}{2} - {\rm i}\sigma_y \sin\frac{\theta}{2} \end{aligned}$$

so that

$$R(\pi, 0) = -{\rm i}\sigma_x \qquad R\Big(\pi, \frac{\pi}{2}\Big) = -{\rm i}\sigma_y$$

We have, for example,

$$\begin{aligned} R\Big(\pi, \frac{\pi}{2}\Big) R(\beta, 0) R\Big(\pi, \frac{\pi}{2}\Big) &= (-{\rm i}\sigma_y)\left(I\cos\frac{\beta}{2} - {\rm i}\sigma_x\sin\frac{\beta}{2}\right)(-{\rm i}\sigma_y) \\ &= -\left(I\cos\frac{\beta}{2} - {\rm i}\sigma_x \sin\frac{\beta}{2}\right) = -R(-\beta, 0) \end{aligned}$$

Let us call A the transition $|0,0\rangle \leftrightarrow |1,1\rangle$ and B the transition $|0,1\rangle \leftrightarrow |1,2\rangle$. The Rabi frequencies are linked by $\omega_B = \sqrt{2}\,\omega_A$. Thus, if the rotation angle is θ_A for transition A, it will be $\theta_B = \sqrt{2}\,\theta_A$ for transition B. For transition A, we choose $\alpha = \pi/\sqrt{2}$ and $\beta = \pi$

$$R\Big(\frac{\pi}{\sqrt{2}}, \frac{\pi}{2}\Big) R(\pi, 0) R\Big(\frac{\pi}{\sqrt{2}}, \frac{\pi}{2}\Big) R(\pi, 0) = -I$$

For transition B we shall have $\alpha = \pi$ and $\beta = \pi\sqrt{2}$

$$R\Big(\pi, \frac{\pi}{2}\Big) R(\pi\sqrt{2}, 0) R\Big(\pi, \frac{\pi}{2}\Big) R(\pi\sqrt{2}, 0) = -I$$

The state $|1,0\rangle$ is not affected because the transition $|0,0\rangle \leftrightarrow |1,0\rangle$ does not resonate on the blue sideband frequency. Thus we have

$$|00\rangle \leftrightarrow -|0,0\rangle \quad |0,1\rangle \leftrightarrow -|0,1\rangle \quad |1,0\rangle \leftrightarrow +|1,0\rangle \quad |1,1\rangle \leftrightarrow -|1,1\rangle$$

5. $R\Big(\pm\pi, \pi/2\Big) = \mp{\rm i}\sigma_y$ so that

$$R\Big(\pm\pi, \frac{\pi}{2}\Big)|0,1\rangle = \mp|1,0\rangle \qquad R\Big(\pm\pi, \frac{\pi}{2}\Big)|1,0\rangle = \pm|0,1\rangle$$

Let us start from the general two ion state, where both ions are in the vibrational ground state

$$\begin{aligned} |\Psi\rangle &= (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) \otimes |0\rangle \\ &= a|00,0\rangle + b|01,0\rangle + c|10,0\rangle + d|11,0\rangle \end{aligned}$$

The action of $R^{-(2)}(-\pi, \pi/2)$ on ion 2 gives

$$|\Psi'\rangle = R^{-(2)}(-\pi, \pi/2)|\Psi\rangle = a|00,0\rangle + b|00,1\rangle + c|10,0\rangle + d|10,1\rangle$$

Then we apply $R^{+(1)}_{\alpha\beta}$ on ion 1

$$|\Psi''\rangle = R^{+(1)}_{\alpha\beta}|\Psi'\rangle = -a|00,0\rangle - b|00,1\rangle + c|10,0\rangle - d|10,1\rangle$$

and finally $R^{-(2)}(\pi,\pi/2)$ on ion 2

$$\begin{aligned} |\Psi'''\rangle = R^{-(2)}(\pi,\pi/2)|\Psi''\rangle &= -a|00,0\rangle - b|01,0\rangle + c|10,0\rangle - d|11,0\rangle \\ &= (-a|00\rangle - b|01\rangle + c|10\rangle - d|11\rangle)\otimes|0\rangle \end{aligned}$$

This is the result of applying a cZ gate, within trivial phase factors.

6.5.4 Vibrational modes of two ions in a trap

Setting $z_1 = z_0 + u$, $z_2 = -z_0 + v$ and expanding to second order in powers of u and v we get

$$V \simeq \frac{1}{2}M\omega_z^2\left(2z_0^2 + 2z_0(u-v) + u^2 + v^2\right) + \frac{e^2}{z_0}\left(1 - \frac{u-v}{2z_0} + \frac{(u-v)^2}{4z_0^2}\right)$$

with $e^2 = q^2/(4\pi\varepsilon_0)$. The equilibrium condition is given by the condition that the terms linear in u and v vanish

$$M\omega_z^2 z_0 - \frac{e^2}{2z_0^2} = 0$$

so that

$$z_0 = \left(\frac{1}{2}\right)^{1/3} l \qquad l = \left(\frac{e^2}{M\omega_z^2}\right)^{1/3}$$

The normal modes are obtained by examining the terms quadratic in u and v, which lead to a potential energy

$$U(u,v) = \frac{1}{2}M\omega_z^2\left(u^2+v^2\right) + \frac{e^2}{4z_0^3}(u-v)^2$$

The equations of motion are

$$\begin{aligned} M\ddot{u} &= -M\omega_z^2 u - \frac{e^2}{2z_0^3}(u-v) = -M\omega_z^2(2u-v) \\ M\ddot{v} &= -M\omega_z^2 v - \frac{e^2}{2z_0^3}(v-u) = -M\omega_z^2(2v-u) \end{aligned}$$

The center of mass mode $(u+v)/2$ oscillates at frequency ω_z

$$(\ddot{u}+\ddot{v}) = -\omega_z^2(u+v)$$

while the breathing mode $(u-v)$ oscillates with frequency $\sqrt{3}\,\omega_z$

$$(\ddot{u}-\ddot{v}) = -3\omega_z^2(u-v)$$

6.5.5 Meissner effect and flux quantization

1. We start from the expression (6.43) of the electromagnetic current

$$\vec{\jmath}_{\rm em} = \frac{\hbar q}{m}\left(\vec{\nabla}\theta(\vec{r}) - \frac{q}{\hbar}\vec{A}(\vec{r})\right)\rho(\vec{r})$$

Let us take the curl of the preceding equation, assuming $\rho(\vec{r})$ to be constant

$$\vec{\nabla}\times\vec{\jmath}_{\rm em} = -\frac{q^2}{m}\rho\vec{B}$$

From the Maxwell equation $\vec{\nabla}\times\vec{B}=\mu_0\vec{\jmath}_{\rm em}$ we also have

$$-\nabla^2\vec{B}=\mu_0\vec{\nabla}\times\vec{\jmath}_{\rm em}$$

and comparing the two equations we obtain

$$\nabla^2\vec{B}==\frac{q^2\rho}{m}\vec{B}=\frac{1}{\lambda_L^2}\vec{B}\qquad\qquad \lambda_L^2=\frac{m}{\mu_0 q^2\rho}=\frac{m_e}{2\mu_0 q_e^2\rho}$$

Taking a one-dimensional geometry, where the region $z>0$ is superconducting, we see that the magnetic field must decrease as

$$B(z)=B(z=0)\mathrm{e}^{-z/\lambda}$$

From $\vec{\nabla}\times\vec{B}=\mu_0\vec{\jmath}_{\rm em}$, we see that the electromagnetic current must also vanish in the bulk of a superconductor.

2. Let us take a ring geometry and draw a contour C well inside the ring. Then we have

$$0=\oint_C\vec{\jmath}_{\rm em}\cdot\mathrm{d}\vec{l}=\frac{\hbar q}{m}\oint_C\vec{\nabla}\theta\cdot\mathrm{d}\vec{l}-\frac{q^2\rho}{m}\oint_C\vec{A}\cdot\mathrm{d}\vec{l}$$

Since $\exp(\mathrm{i}\theta)$ is single valued, we must have $\theta\to\theta+2\pi n$ after a full turn, and

$$\frac{\hbar q}{m}(2\pi n)=\frac{q^2\rho}{m}\int\!\!\int\vec{B}\cdot\mathrm{d}\vec{S}\qquad n=\cdots,-1,0,1,2,\cdots$$

6.5.6 Josephson current

Let us start from (6.45) and write

$$\psi_i=\rho_i\mathrm{e}^{\mathrm{i}\theta_i}\qquad\qquad i=1,2$$

The first of the equations (6.45) becomes

$$\frac{\mathrm{i}\hbar}{2}\frac{\mathrm{d}\rho_1}{\mathrm{d}t}-\hbar\rho_1\frac{\mathrm{d}\theta_1}{\mathrm{d}t}=\frac{1}{2}qV\rho_1+K\sqrt{\rho_1\rho_2}\,\mathrm{e}^{\mathrm{i}\theta}$$

with $\theta=\theta_2-\theta_1$. Taking the real and imaginary parts of this equation and the corresponding equation for $i=2$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\rho_1}{\mathrm{d}t}&=\frac{2K}{\hbar}(\rho_1\rho_2)^{1/2}\sin\theta,\\ \frac{\mathrm{d}\rho_2}{\mathrm{d}t}&=-\frac{2K}{\hbar}(\rho_1\rho_2)^{1/2}\sin\theta,\\ \frac{\mathrm{d}\theta_1}{\mathrm{d}t}&=-\frac{K}{\hbar}\left(\frac{\rho_2}{\rho_1}\right)^{1/2}\cos\theta-\frac{q_cV}{2\hbar},\\ \frac{\mathrm{d}\theta_2}{\mathrm{d}t}&=\frac{K}{\hbar}\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\cos\theta+\frac{q_cV}{2\hbar}\end{aligned}$$

and subtracting the last but one equation from the last one

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{q_cV}{\hbar}$$

6.5.7 Charge qubits

From the relation

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\langle n|\theta\rangle\langle\theta|m\rangle=\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{-\mathrm{i}(n-m)\theta}=\delta_{nm}$$

we derive

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, |\theta\rangle\langle\theta| = I$$

Furthermore

$$N|\theta\rangle = \sum_n n\mathrm{e}^{-\mathrm{i}n\theta}|n\rangle = \mathrm{i}\frac{\partial}{\partial\theta}\left(\sum_n \mathrm{e}^{-\mathrm{i}n\theta}|n\rangle\right)$$

so that

$$N = \mathrm{i}\frac{\partial}{\partial\theta}$$

We can also use the commutation relation

$$[N, \Theta] = \mathrm{i}I$$

to obtain

$$\mathrm{e}^{-\mathrm{i}\Theta}\, N\, \mathrm{e}^{\mathrm{i}\Theta} = N - \mathrm{i}[\Theta, N] = N - I$$

and to derive

$$N\left(\mathrm{e}^{\mathrm{i}\Theta}|n\rangle\right) = \mathrm{e}^{\mathrm{i}\Theta}(N - I)|n\rangle = (n-1)\left(\mathrm{e}^{\mathrm{i}\Theta}|n\rangle\right)$$

We may then choose the phases of the states $|n\rangle$ such that

$$\mathrm{e}^{\mathrm{i}\Theta}|n\rangle = |n-1\rangle \qquad \mathrm{e}^{-\mathrm{i}\Theta}|n\rangle = |n+1\rangle$$

and thus

$$\cos\Theta|n\rangle = \frac{1}{2}\left(|n-1\rangle + |n+1\rangle\right)$$

2. In the vicinity of $n_g = 1/2$, the Hamiltonian becomes

$$\hat{H} =\simeq \frac{1}{4}\, E_c I + E_c\left(n_g - \frac{1}{2}\right)|0\rangle\langle 0| - E_c\left(n_g - \frac{1}{2}\right)|1\rangle\langle 1| - \frac{1}{2}\, E_J\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)$$

In the $\{|0\rangle, |1\rangle\}$ basis, this can be written, omitting the (irrelevant) constant term

$$\hat{H} \simeq E_c\left(n_g - \frac{1}{2}\right)\sigma_z - \frac{1}{2}\, E_J\sigma_x$$

If n_g is far enough from 1/2, the eigenvectors of $\hat{H}$ are approximately the vectors $|0\rangle$ and $1\rangle$, due to the condition $E_c \gg E_J$. When n_g comes close to 1/2, tunneling becomes important, and at $n_g = 1/2$, the eigenvectors are those $|\pm\rangle$ of σ_x with eigenvalues $\pm E_J$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \qquad \sigma_x|\pm\rangle = \pm|\pm\rangle$$

One observes the standard phenomenon of level repulsion around $n_g = 1/2$. It is usual to exchange the x and z bases, so that the control parameter appears as the coefficient of σ_x

$$\hat{H} \simeq -\frac{1}{2}\, E_J\sigma_z + E_c\left(n_g - \frac{1}{2}\right)\sigma_x$$

Chapter 7

Exercises from Chapter 7

7.5.1 Superdense coding

1. From the identities

$$\sigma_x|0\rangle = |1\rangle \quad \sigma_x|1\rangle = |0\rangle \quad \sigma_z|0\rangle = |0\rangle \quad \sigma_z|1\rangle = -|1\rangle$$

we immediately get

$$\begin{aligned}
A_{00}|\Psi\rangle &= |\Psi\rangle \\
A_{01}|\Psi\rangle &= \frac{1}{\sqrt{2}}\big(|0_A \otimes 0_B\rangle - |1_A \otimes 1_B\rangle\big) \\
A_{10}|\Psi\rangle &= \frac{1}{\sqrt{2}}\big(|1_A \otimes 0_B\rangle + |0_A \otimes 1_B\rangle\big) \\
A_{11}|\Psi\rangle &= \frac{1}{\sqrt{2}}\big(|1_A \otimes 0_B\rangle - |0_A \otimes 1_B\rangle\big)
\end{aligned}$$

Let us first examine the action of the cNOT-gate on the four states $A_{ij}|\Psi\rangle$

$$\begin{aligned}
\text{cNOT}\big[A_{00}|\Psi\rangle\big] &= \frac{1}{\sqrt{2}}\big(|0_A\rangle + |1_A\rangle\big) \otimes |0_B\rangle \\
\text{cNOT}\big[A_{01}|\Psi\rangle\big] &= \frac{1}{\sqrt{2}}\big(|0_A\rangle - |1_A\rangle\big) \otimes |0_B\rangle \\
\text{cNOT}\big[A_{10}|\Psi\rangle\big] &= \frac{1}{\sqrt{2}}\big(|0_A\rangle + |1_A\rangle\big) \otimes |1_B\rangle \\
\text{cNOT}\big[A_{11}|\Psi\rangle\big] &= -\frac{1}{\sqrt{2}}\big(|0_A\rangle - |1_A\rangle\big) \otimes |1_B\rangle
\end{aligned}$$

The measurement of qubit B has the result $|0_B\rangle$ for $i = 0$ (A_{00} and A_{01}) and $|1_B\rangle$ for $i = 1$ (A_{10} and A_{11}), so that this measurement gives the value of i. Furthermore

$$\begin{aligned}
H\frac{1}{\sqrt{2}}\big(|0_A\rangle + |1_A\rangle\big) &= |0_A\rangle \\
H\frac{1}{\sqrt{2}}\big(|0_A\rangle - |1_A\rangle\big) &= |1_A\rangle
\end{aligned}$$

and the measurement of qubit A gives the value of j.

7.5.2 Shannnon entropy versus von Neumann entropy

The state matrix ρ is given by

$$\rho = \begin{pmatrix} \mathsf{p} + (1-\mathsf{p})\cos^2\theta/2 & (1-\mathsf{p})\sin\theta/2\,\cos\theta/2 \\ (1-\mathsf{p})\sin\theta/2\,\cos\theta/2 & (1-\mathsf{p})\sin^2\theta/2 \end{pmatrix}$$

and its eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\mathsf{p}(1-\mathsf{p})\sin^2\theta/2}\right) = \frac{1}{2}(1 \pm x)$$

This allows us to write the von Neumann entropy as (it is convenient to use ln rather than log)

$$-H_{\rm vN} = \frac{1+x}{2}\,\ln\frac{1+x}{2} + \frac{1-x}{2}\,\ln\frac{1-x}{2}$$

Let us compute the x-derivative of $H_{\rm vN}$

$$-\frac{\mathrm{d}}{\mathrm{d}x}H_{\rm vN}(x) = \frac{1}{2}\ln\frac{1+x}{1-x} = \tanh^{-1}(x)$$

Thus $H_{\rm vN}(x)$ is a concave function of x which has a maximum at $x=0$: $H_{\rm vN}(x=0) = \ln 2$. For this value of x, we have $H_{\rm vN} = H_{\rm Sh}$. Let us write $\mathsf{p} = (1+\overline{\mathsf{p}})/2$, so that

$$H_{\rm Sh} = \frac{1+\overline{\mathsf{p}}}{2}\,\ln\frac{1+\overline{\mathsf{p}}}{2} + \frac{1-\overline{\mathsf{p}}}{2}\,\ln\frac{1-\overline{\mathsf{p}}}{2}$$

and

$$x = \sqrt{1-(1-\overline{\mathsf{p}})^2\sin^2\theta/2}$$

Now we have the inequality

$$\overline{\mathsf{p}}^2 - x^2 = -(1-\overline{\mathsf{p}}^2)\cos^2\theta/2 \le 0$$

Thus $|x| \ge \overline{\mathsf{p}}$ and $H_{\rm Sh} \ge H_{\rm vN}$.

7.5.3 Information gain of Eve

1. Alice uses the bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$, while Eve uses the basis $\{|0\rangle, |1\rangle\}$. The conditional probabilities $\mathsf{p}(r|i)$ are

$$\mathsf{p}(0|0) = 1 \quad \mathsf{p}(1|0) = 0 \quad \mathsf{p}(0|1) = 0 \quad \mathsf{p}(1|1) = 1$$

and

$$\mathsf{p}(0|+) = 1/2 \quad \mathsf{p}(1|+) = 1/2 \quad \mathsf{p}(0|-) = 1/2 \quad \mathsf{p}(1|-) = 1/2$$

We obtain $\mathsf{p}(r)$ from

$$\mathsf{p}(r) = \sum_i \mathsf{p}(r|i)\mathsf{p}(i) = \frac{1}{2} \quad \forall r$$

Let us now turn to the conditional probabilities $\mathsf{p}(i||r)$ (we use a double vertical bar to underline the difference with $\mathsf{p}(r|i)$)

$$\mathsf{p}(i||r) = \frac{\mathsf{p}(r|i)\mathsf{p}(i)}{\mathsf{p}(r)} = \frac{1}{2}\,\mathsf{p}(r|i)$$

We find

$$\begin{aligned} \mathsf{p}(0||0) &= \frac{1}{2} \quad \mathsf{p}(1||0) = 0 \quad \mathsf{p}(+||0) = \frac{1}{4} \quad \mathsf{p}(-||0) = \frac{1}{4} \\ \mathsf{p}(1||0) &= 0 \quad \mathsf{p}(1||1) = \frac{1}{2} \quad \mathsf{p}(+||1) = \frac{1}{4} \quad \mathsf{p}(-||1) = \frac{1}{4} \end{aligned}$$

Before Eve's measurement, the (Shannon) entropy is $H(\alpha) = 2$, after Eve's measurement, the entropy $H(\alpha|\varepsilon)$ is, from (7.10)

$$\begin{aligned} H(\alpha|\varepsilon) &= -\sum_r \mathsf{p}(r)\,\sum_i \mathsf{p}(i||r)\log\mathsf{p}(i||r) \\ &= -\frac{1}{2}\,\log\frac{1}{2} - 2\left(\frac{1}{4}\,\log\frac{1}{4}\right) = \frac{3}{2} \end{aligned}$$

The information gain of Eve is

$$I(\alpha : \varepsilon) = H(\alpha) - H(\alpha|\varepsilon) = \frac{1}{2}$$

2. Eve uses a $\{|+\pi/8\rangle, |-\pi/8\rangle\}$ basis, so that $\mathsf{p}(r|i)$ is given by

$$\begin{aligned} \mathsf{p}(0|1) &= \mathsf{p}(0|+) = .854 \qquad \mathsf{p}(1|0) = \mathsf{p}(1|-) = 0.146 \\ \mathsf{p}(1|1) &= \mathsf{p}(1|+) = .146 \qquad \mathsf{p}(1|1) = \mathsf{p}(1|-) = 0.854 \end{aligned}$$

We then compute $\mathsf{p}(i||r)$

$$\mathsf{p}(0||0) = \mathsf{p}(+||0) = 0.427 \qquad \mathsf{p}(1||0) = \mathsf{p}(-||0) = .073$$

and

$$H(\alpha|\varepsilon) = -\sum_r \mathsf{p}(r) \sum_i \mathsf{p}(i||r) \log \mathsf{p}(i||r) = 1.600$$

so that the information gain is now

$$I(\alpha : \varepsilon) = 0.400$$

7.5.4 Symmetry of the fidelity

Let us take $\rho = |\Psi\rangle\langle\Psi|$, a pure state, and write σ as

$$\sigma = \sum_\alpha \mathsf{p}_\alpha |\alpha\rangle\langle\alpha|$$

Observing that $\rho^{1/2} = |\Psi\rangle\langle\Psi|$, we obtain

$$\rho^{1/2}\sigma\rho^{1/2} = |\Psi\rangle\langle\Psi|\langle\Psi|\sigma|\Psi\rangle$$

from which it follows that

$$\mathcal{F}(\rho, \sigma) = \langle\Psi|\sigma|\Psi\rangle$$

Let us now take $\sigma = |\Psi\rangle\langle\Psi|$ and use the diagonal form of ρ

$$\rho = \sum_i \mathsf{p}_i |i\rangle\langle i|$$

Since $\sigma = |\Psi\rangle\langle\Psi|$ is a rank one operator, the same is true for $(\rho^{1/2}\sigma\rho^{1/2})$ whose eigenvalue equation is, in a space of dimension N

$$\lambda^N - \mathrm{Tr}(\rho^{1/2}\sigma\rho^{1/2})\,\lambda^{N-1} = 0$$

whence

$$\lambda = \mathrm{Tr}\,(\rho^{1/2}\sigma\rho^{1/2})$$

is the only non zero eigenvalue. Let us decompose $|\Psi\rangle$ on the $\{|i\rangle\}$ basis

$$|\Psi\rangle = \sum_i c_i |i\rangle$$

and write the matrix $\rho^{1/2}\sigma\rho^{1/2}$ in this basis

$$(\rho^{1/2}\sigma\rho^{1/2})_{ij} = \sqrt{\mathsf{p}_i \mathsf{p}_j}\, c_i c_j^*$$

From this we deduce

$$\mathrm{Tr}\,(\rho^{1/2}\sigma\rho^{1/2}) = \sum_i \mathsf{p}_i |c_i|^2 = \langle\Psi|\rho|\Psi\rangle = \lambda$$

and

$$\left(\mathrm{Tr}\,\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right)^2 = \lambda = \langle\Psi|\rho|\Psi\rangle = \mathcal{F}(\sigma, \rho)$$

7.5.5 Quantum error correcting code

From $X|\pm\rangle = \pm|\pm\rangle$, we obtain, for example

$$\begin{aligned} X_A X_B|\Psi_A\rangle &= X_A X_B(\lambda|-++\rangle + \mu|+--\rangle) = -|\Psi_A\rangle \\ X_A X_C|\Psi_A\rangle &= X_A X_C(\lambda|-++\rangle + \mu|+--\rangle) = -|\Psi_A\rangle \end{aligned}$$

Let us also check

$$\begin{aligned} \mathrm{cNOT}_B \mathrm{cNOT}_C (H_A \otimes H_B \otimes H_C)|\Psi_C\rangle &= \mathrm{cNOT}_B \mathrm{cNOT}_C(\lambda|001\rangle + \mu|110\rangle) \\ &= \lambda|001\rangle + \mu|101\rangle) = (\lambda|0\rangle + \mu|1\rangle) \otimes |01\rangle \end{aligned}$$